Optimization in Statistics

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What is optimization?

- Have actually already seen several examples of optimization in this class.
- In statistics we try to fit models which approximate our data.
- Want to choose the model which "best" approximates our data.
- Want to "optimize" over a selection of models to find the "best" one.

Linear Regression

- In this setting we have the model

  $$y = \alpha + \beta x,$$

  and we wanted to find the best pair $\hat{\alpha}, \hat{\beta}$ which best fit our data. This is a simple example of optimization.

A quick review of Calculus

- If we differentiate a function and find the values where this derivative are zero, these are turning points of the function.
- We establish if these are local maxima or local minima by evaluating the second derivative at these turning points.
- For certain types of functions (convex/concave), then these local optima may be global.

Finding the max/min of a function

- Write this down
- When fitting a model, we want to come up with a function which describes the difference between our data and the model, and minimize this function.
- This is often known as the loss function.
- Some examples of loss functions are:

Global Max

- Certain types of functions have only one maximum, and it can be found in a straightforward way, using ...

Newton-Rhapson method

- Want to find roots of some function $f(x)$.
- Start with some inital estimate x_0
- Improve this estimate iteratively with the formula,

  $$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

  until it changes only by a small amount.
- Once you start suitably close to the zero you will reach it.

More general methods

- Gradient descent/etc

Functions without a unique maximum

- Many common functions are more complex and do not have a global maximum, instead several (or possibly infinite) local maximums.
- Can be difficult (or impossible) to find which of these maximums is the global maximum.
- When optimizing high dimensional functions this can be challenging.

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- No guarantee this will work, but as we will see, is often all that can be used.

Clustering

- Clustering is an extremely common method in statistics and data science.
- It is an unsupervised learning problem. Given some data, we want to try find clusters in the data which reveal interesting relationships.
- This is different to classification, where there are some known labels and we want to predict these labels for some new data.

K-means

- K-means clustering is one of the most common clustering methods.
- It aims to partition data into k groups. Each of the k groups has a center, and a data point is assigned to the cluster corresponding to the nearest center.
- The algorithm tries to find centers which create clusters which are close together, and classifies points to the corresponding closest center.

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- This looks tricky to solve, and it looks like it might not have a global maximum.

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- How could one go about doing this?

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- Update the cluster centers, then update the cluster each point is placed in.
- Then repeat this many times until the clusters stop changing.
- There is generally no theoretical reason for this to work but does in practice.

Other optimization methods

- There are lots of more advanced methods to optimize functions.
- To be brief, they all do gradient descent, or some slight variant of it.